Elektrichestvo (Electricity), 2016, № 1, c. 35–43.

Keller–Dykhne method of inversion to define integral parameters of multi-electrode two-dimensional systems[1]

KLYAMKIN S.S. *(Vernon, California, USA) — Scientist*

A new method to calculate integral parameters for two-dimensional multi-electrode structures having symmetry axes or planes (conductivity, capacitance etc.) has been proposed. The method is based on a simultaneous solution of algebraic equations for original and inverted systems (structures) connected between them according to the principle of rotatory invariance of two-dimensional potential and solenoidal fields. Then unknown formulas obtained on the basis of this method for calculation of linear capacitance of a series of systems have been compiled in table showed in the Appendix. The method makes it possible to extend the scope of application of Thompson–Lampard theorem on mutual partial capacitance per unit length in a system containing four cylindrical conducting plates with symmetry planes. New solutions found on the basis of Thompson–Lampard theorem may be used to solve a wider range of problems in electric engineering. The connection between exact solutions of symmetric multi-electrode systems and geometric parameters of regular polygons enables us to study the properties of similar systems without performing additional calculations. The method may be used for calculation of not only linear capacitance but other integral parameters of two-dimensional physical systems under different conditions of symmetry and medium features distribution.

Key words: *multi-electrode structures, conductivity, capacitance, two-dimensional physical systems, calculate integral parameters, Thompson—Lampard theorem*

Keller's theorem [1] belongs to the theory of two-dimensional heterogeneous media, namely to the evaluation of average ("efficient") specific conductivity of these media in two mutually orthogonal directions. The other fundamental paper concerning the same subject is the article of Dykhne [2] where similar results have been obtained on the basis of clearly more demonstrative differential definition of rotational invariance principle for two-dimensional field. The backbone of Dykhne's procedure just as of the Keller's one is that a potential two-dimensional force field and a solenoidal flow field connected to it by Maxwell equation exchange their roles when rotated by 90° in each point; when a dimensionless specific conductivity (s, e, m...) is replaced in each point by inverted value (1/s, 1/e, 1/m ...) they represent new potential and solenoidal fields connected by Maxwell equation.

The similarity of solutions of two-dimensional inverse (in this context) problems has been always observed. However, a conjoint solution of mutually inverse problems by Keller and Dykhne introduced new analytical expressions for averaged parameters of heterogeneous media and in case of a certain symmetry (regularity) of these media allowed to obtain previously unknown exact formulae for averaged specific conductivity and some integral characteristics for each

[1] The author thanks Professor V. L. Chechurin for friendly support and laborious contribution in the preparation of this article.

of two mutually inverse media. That solution appears to become a new successful method that has been generalized and developed in a large number of published papers of other authors.

Thus, there have been found exact expressions of averaged specific conductivity ye of two-dimensional two-component symmetric structure with equal concentration of generating components. Specifically, for field quincunx or random highly dispersed structures $\sigma_e = \sqrt{\sigma_1 \sigma_2}$, where σ_1 and σ_2 are specific conductivities of generating components (in field quincunx structure — black and white checker squares).

To be specific, all mentioned hereafter solutions shall be referred to as electrostatic problems.

We shall call this method as Keller-Dykhne inversion method. The numerous problems related both to the evaluation of efficient dielectric permeability of regular composite media (heterogeneous, anisotropic) and integral parameters of some electrode systems in heterogeneous medium (mutual partial capacitances etc.) may be solved with the aid of this method.

Previously, there have been solved some similar problems for two-electrode systems with the help of the above-mentioned method [3, 4, 5].

The goal of this paper is to develop the Keller—Dykhne inversion method and to get new solutions for multi-electrode systems. We shall study electrically neutral heterogeneous two-dimensional cylindrical systems containing *n* electrodes separated by *n* impermeable membranes (boundaries) and forming in cross-section together the closed contours encircling simply connected flat surfaces.

Solution method. The method is intended to define integral parameters of two mutually inverted two-dimensional electrically neutral *n*-electrode systems by generating and conjoint solving 2*n* linear algebraic equations for charges on electrodes and electrode-to-electrode voltages.

The method of solution is based on the procedure of matched field vector rotation, on the inversion of dielectric permeability values and on setting relevant numerical values of unlike charges and voltages of initial system equal to unlike voltages and charges of inversed system.

It is believed that all initial mutual partial capacitances are unknown.

In this case their values as unique solution of a given system of equations may be obtained only for electrode systems having a particular symmetry conditioning a certain equality of these parameters in mutually inverse systems and reducing thereby the number of required unknown parameters. A formal procedure for comparing mutual partial capacitances in inverse systems for purposes of identifying this equality means to superpose (match) inverse systems by their relative non-deformable transfer or rotation. A desired condition specifying required equations of mutual partial capacitances is a full matching of mutually inverted systems (congruence of limiting contours and matching of dielectric permeability in matching points, including boundary points those are conducting for electrodes and non-conducting for impermeable boundaries).

For the purpose of solution clearness the following has been used:

dimensionless (numerical) equations of unlike values using the equating sign «$\doteq$»;

figures exhibiting the procedure of matched field rotation and mutual transformation of charges and voltages common for mutual systems;

directional images (arrow) of scalar electrode charges and electrode-to-electrode voltages presenting their signs;

normalization of dielectric permeability of heterogeneous medium ensuring an interchangeability of some symmetrical parts or points of heterogeneous medium at their inversion (i.e. equal to 1 product of normalized permeabilities in symmetrical points for a sensitive indicative case of the matched mutual systems).

Let us give an example of dielectric permeability normalization. Let a two-component medium have arbitrary values of components ε_1 and ε_2 permeability. We shall replace this medium by another one having the permeability of the same components

$$\varepsilon'_1 \doteq \varepsilon_1 / \sqrt{\varepsilon_1 \varepsilon_2}; \ \varepsilon'_2 \doteq \varepsilon_2 / \sqrt{\varepsilon_1 \varepsilon_2}.$$

We shall have as a result a new medium where the component permeability is mutually invertible, i.e. $\varepsilon'_1 \varepsilon'_2 \doteq 1$, and all integral parameters of original system are connected to the parameters of normalized system by common dimensionless proportionality factor $K = \sqrt{\varepsilon_1 \varepsilon_2}$.

A track in the plane of intersection of dimensionless cylinder (perpendicular transverse section) is presented in Fig. 1 on the left. The cylinder contains alternant electrodes *1, 2* and impermeable boundaries *a, b* being in isotropic homogeneous medium with dielectric permeability ε. Electrodes *1* and *2* are charged uniformly along cylinder length with full charge on unit length surface (charge per unit length) *Q* and (–*Q*) respectively. Electrode-to-electrode voltage is *U*. For vectors of electric displacement $\boldsymbol{D}$ and electric field strength $\boldsymbol{E}$ in any point on the plane, including boundary, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$. Vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ are located everywhere in the plane of section and they are directed on electrodes at right angle to their surface and on impermeable membranes tangentially to their surface. Specifically, let us consider an interior of cylinder. On positively charged electrode *1* field vectors are directed away from the electrode (charge Q marked with arrow corresponds to them), on negatively charged electrode *2* – towards it. On membranes a and b field vectors D and E are directed from electrode *1* to electrode *2* (voltage U is directed backward), as the potential of positively charged electrode *1* is higher than that of electrode *2.* By definition, a unit-length mutual partial capacitance between electrodes is $C_{12} = Q / U$.

Matched rotation of all vectors in each point by 90° in planes normal to cylinder axis (specifically – counterclockwise rotation) and simultaneous inversion of dielectric permeability together transform an original system and field into mutual one presented on Fig. 1 at the right with electrodes *a, b* and impermeable boundaries *1, 2.*

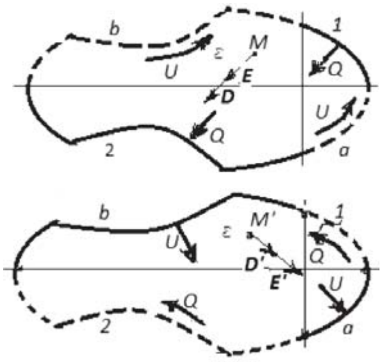


Fig. 1

Charges and voltages on both mutual inverted systems are shown by arrows (assumed conventional direction with respect to electrode surface – from positive to negative and electrode-to-electrode potential drop - from smaller to larger correspond to the sign of scalar quantities of charges and voltages). These arrows represent at the same time average values of vectors $\boldsymbol{D}$ and $(-\boldsymbol{E})$ on respective boundary surfaces. It should be noted that to matched counterclockwise rotation of all field vectors $\boldsymbol{D}$ and $\boldsymbol{E}$, as it is shown in Figure, corresponds a matched clockwise rotation of all charge and voltage arrows (this difference is a consequence of arrows opposite direction showing the field strength from positive to negative potentials and the voltage between two points – from negative to positive potentials).

So, in a dual problem, according to the vector rotation, electrodes and impermeable membranes exchange their places at boundaries, and, while keeping their value in modulus, electrode-to-electrode voltages become charges on electrodes and vice versa. Thus, given the shape of transverse section is arbitrary, a mutual partial capacitance for dual problem is

$$C_{ab=} U / Q = 1 / C_{12}, \text{ or } C_{12} C_{ab=} 1.$$

If $\text{e}_{=} 1$ and the section has an axis of anti-symmetry, then $C_{12} = C_{ab=} 1$; thus for non-normalized problem, provided the normalized mutual systems are matching

$$C_{12} = C_{ab} = \sqrt{\text{e}_M \text{e}_N}.$$

A common solution of 3-electrode system shown below, may be considered as a typical model for calculation of electrically neutral multi-electrode system with piecewise homogeneous medium.

Transverse sections of two mutual inverse 3-electrode systems built in the same way as in the previous problem are shown in Fig. 2.

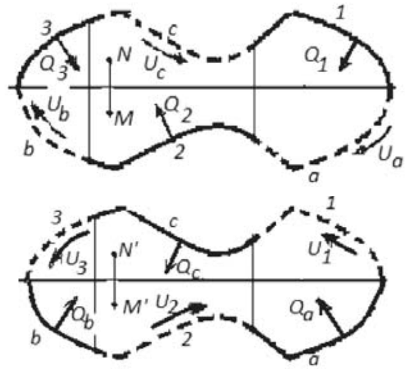


Fig. 2

The product of dielectric permeability of each system for any pair of antisymmetric points (for example, M and N) is equal to *1*, i.e. systems are normalized.

A plane reflection of one of the systems with respect to its anti-symmetry axis is congruent and fully matching with the other system that is mutual to it (matching electrodes, membranes, dielectric permeability and dimensionless fields). Therefore the following relation holds true: $Q_{k=} U_k$, where k = 1, 2, 3, *a, b, c*; $C_{12} = C_{ac}$, $C_{13} = C_{ab}$, $C_{23} = C_{bc}$. Conditions of electro neutrality are met at: $Q_1 + Q_2 + Q_3 = 0$, $Q_a + Q_b + Q_c = 0$, $U_a + U_b + U_c = 0$, $U_1 + U_2 + U_3 = 0$.

A system of equations connecting charges, voltages and mutual partial capacitances of both electrode systems has the following form:

$$Q_1 = C_{12}(- U_a) + C_{13} U_c;\ Q_2 = C_{12} U_a - C_{23} U_b;$$

$$Q_3 = C_{13}(- U_c) + C_{23} U_b;\ Q_a = C_{ac}(- U_1) + C_{ab} U_2;$$

$$Q_b = C_{ab}(- U_2) + C_{bc} U_3;\ Q_c = C_{ac} U_1 + C_{bc} U_3.$$

As $Q_{k=} U_k$, charges and voltages of one of the electrode systems may be excluded, so, we shall have the following for numerical values of all quantities making part of equations describing the first electrode system

$$Q_{1=} C_{12}(- U_a) + C_{13} U_c;\ Q_{2=} C_{12} U_a - C_{23} U_b;$$

$$Q_{3=} C_{13}(- U_c) + C_{23} U_b;\ U_{a=} C_{12}(- Q_1) + C_{13} Q_2;$$

$$U_{b=} C_{13}(- Q_2) + C_{23} Q_3;\ U_{c=} C_{12} Q_1 + C_{23}(Q_3).$$

The relation $C_{12} = C_{13} = C_{23} = C$ follows out of the cyclic symmetry of system equations and uniqueness of solution.

Then, for example, leaving out all charges and setting the voltages to any values that are physically meaningful and meeting the condition of electro neutrality (e.g. $U_a = 1$, $U_b = -1$, $U_c = 0$), we shall have one quadratic equation $3C^2 = 1$ for all mutual partial capacitances of both mutual electrode systems, with a unique positive real solution of it being $C = 1/\sqrt{3}$.

With the increase of the number of electrodes, the number of various (non-coinciding in value) mutual partial capacitances of each electrode system increases, along with the number of quadratic equations to be solved jointly.

Equations may take various forms. One of equation options for fully matching mutual systems at n= 4, 5, 6, 7 is given below.

Equation system for 4 electrodes:

$$3C_{12}^2 + C_{13}^2 + 2C_{12}C_{13} = 1;\ 2C_{12}^2 + 2C_{13}^2 + 4C_{12}C_{13} = 1.$$

Equation system for 5 electrodes:

$$3C_{12}^2 + 3C_{13}^2 + 4C_{12}C_{13} = 1;\ 2C_{12}^2 + 7C_{13}^2 + 6C_{12}C_{13} = 1.$$

Equation system for 6 electrodes:

$$3C_{12}^2 + 3C_{13}^2 + C_{14}^4 + 4C_{12}C_{13} + 2C_{12}C_{14} + 2C_{13}C_{14} = 1;$$

$$2C_{12}^2 + 6C_{13}^2 + 2C_{14}^2 + 6C_{12}C_{13} + 2C_{12}C_{14} + 6C_{13}C_{14} = 1;$$

$$3C_{12}^2 + 3C_{13}^2 + 2C_{14}^2 + 2C_{12}C_{13} + 4C_{12}C_{14} + 4C_{13}C_{14} = 1,$$

Equation system for 7 electrodes:

$$3C_{12}^2 + 3C_{13}^2 + 3C_{14}^4 + 4C_{12}C_{13} + 4C_{12}C_{14} + 4C_{13}C_{14} = 1;$$

$$2C_{12}^2 + 6C_{13}^2 + 7C_{14}^2 + 6C_{12}C_{13} + 4C_{12}C_{14} + 10C_{13}C_{14} = 1;$$

$$3C_{12}^2 + 2C_{13}^2 + 7C_{14}^2 + 2C_{12}C_{13} + 6C_{12}C_{14} + 8C_{13}C_{14} = 1.$$

Expressions for capacitances presented hereinafter represent solutions of specified equation systems at $K = 1$, which is confirmed by direct substitution. Finding analytical solutions (in radicals), if such exist, requires relatively cumbersome algebraic transformations. The search for solution can be significantly facilitated by analysis of special symmetrical excitation modes of respective multi-electrode systems, such as relations between charges and/or voltages that allow to reduce the number of unknown variables and decrease the order of equation system.

Mutual partial capacitances of matching inverse systems. Further on, we shall consider several problems for cylindrical electroneutral two-dimensional electrode systems, that, together with impermeable boundaries, form simply connected closed curves. All results have been received on the basis of Keller-Dykhne inversion and are true for both interior and exterior domains.

1*a*. Two electrodes in homogeneous medium. The system has at least one axis of reflection anti-symmetry, i.e. at dielectric permeability $\varepsilon = 1$ it is matching with a mutual system by rotating around anti-symmetry axis by 180° and parallel transfer.

In Fig. 3,*a*: *1* and *2* — electrodes, *a* and *b* — impermeable boundaries, $\varepsilon = 1$ — cylinder dielectric permeability.

The inversion system of Fig. 3,*a* is shown in Fig. 3,*b*, with dielectric permeability $\varepsilon' = 1/\varepsilon = 1$.

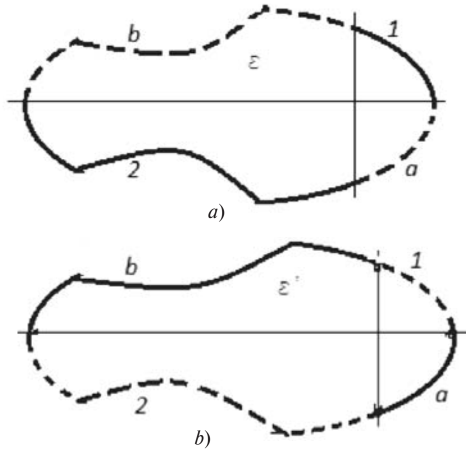


Fig. 3

Unit-length partial mutual capacitances are $C_{12} = C_{ab} = \varepsilon$ at any value of ε for both systems.

A more complicated solution for this problem is known and is quoted in [6] with source references.

1*b*. Two electrodes in homogeneous medium. The system has a center of 2-fold rotational symmetry. A homogeneous system boundary (here and after — conventional unbroken homogeneous closed line coinciding with the boundary of system) has a center of 4-fold rotational symmetry. The system is matching with mutual system being rotated by 90° at $\varepsilon = 1$.

An example of mutual inverse pair of feasible systems [5] is given in Fig. 4,*a, b*.

Unit-length partial mutual capacitance $C_{12} = \varepsilon$.

2. Two electrodes in piecewise or continuously heterogeneous medium. The system is matching with mutual system. The product of dielectric permeabilities of any pair of anti-symmetry points (points *M* and *N*, coinciding with mutual points at superposition of one

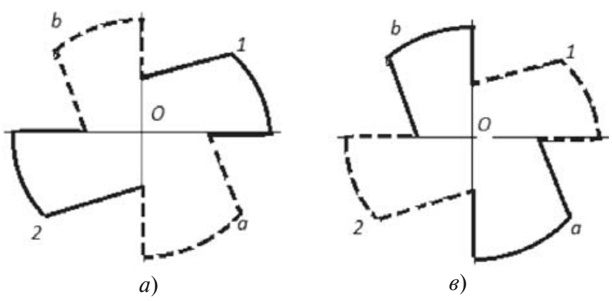


Fig. 4

mutual system on another) is equal to 1 ($\varepsilon_M \varepsilon_N = 1$). The system has at least one axis of anti-symmetry and/or center of 2-fold rotational symmetry. A homogeneous system boundary has an anti-symmetry axis and/or 4-fold center of rotational symmetry.

Examples of respective piecewise homogeneous two-component systems [3, 4, 5] are given in Fig. 5,*a, b*; *M* and *N* — mutual points; grey domains have a dielectric permeability ε_M, white — ε_N.

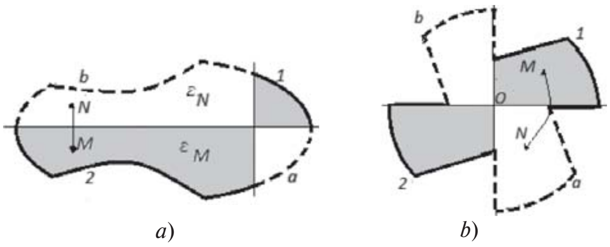


Fig. 5

Unit-length partial mutual capacitance $C_{12} = \sqrt{\varepsilon_M \varepsilon_N} = 1$.

If $\sqrt{\varepsilon_M \varepsilon_N} = K$ (*K* — real number same for any pair of mutual points), then the specified system is reduced (normalized) to a system matching with a mutual one by division of dielectric permeability by *K* in each point, so the unit-length partial mutual capacitance is

$$C_{12} = \sqrt{\varepsilon_M \varepsilon_N} = K.$$

A similar procedure of system reduction to a matching mutual system has been used in studied cases of multi-electrode systems with heterogeneous media.

New results are given in pp. 37—39.

3. Three electrodes in heterogeneous medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$.

The system has at least one axis of anti-symmetry and/or a center of anti-symmetry with respect to its rotation by 180° and/or a center of threefold rotation symmetry. A homogeneous system boundary has a symmetry axis and/or a center of 2- or 3-fold or 6-fold rotational symmetry.

Examples of respective two-component systems are shown in Fig. 6,*a, b, c, d*; white domains have a dielectric permeability ε_M and grey domains ε_N.

A unit-length partial mutual capacitance (analytic solution of a system of three linear equations set for two mutual electrode systems) is equal to

$$C_{12} = C_{23} = C_{13} = \sqrt{\varepsilon_M \varepsilon_N / 3} = K / \sqrt{3}.$$

It should be noted that the equality of all mutual partial capacitances are not be evident following the study of only one of mutual systems as in the case of Fig. 6,*a* and 6,*b*. The same feature is exhibited by some

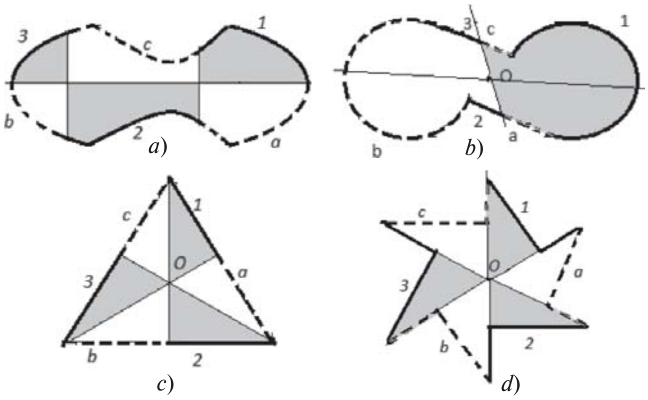


Fig. 6

systems studied below with the number of electrodes exceeding three.

A homogeneous medium represents a special case $\varepsilon_M = \varepsilon_N$.

4. Four electrodes in heterogeneous medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$.

The system has two inter-perpendicular axes of anti-symmetry and/or fourfold center of rotation symmetry. A homogeneous boundary has two inter-perpendicular symmetry axes and/or eightfold center of rotational symmetry.

Examples of respective two-component systems are given in Fig. 7,*a, b*; white domains have a dielectric permeability ε_M and grey domains — ε_N.

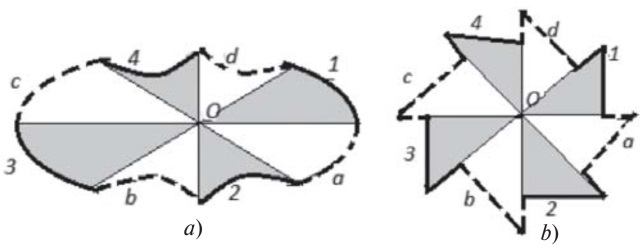


Fig. 7

Unit-length partial mutual capacitances (analytic solution of a system of four linear equations):

$$C_{12} = C_{23} = C_{34} = C_{14} = \sqrt{\varepsilon_M \varepsilon_N} / 2 = K / 2;$$

$$C_{13} = C_{24} = (\sqrt{2} - 1)\sqrt{\varepsilon_M \varepsilon_N} / 2 = K(\sqrt{2} - 1) / 2.$$

Equality $C_{12} = C_{23}$ and $C_{13} = C_{24}$ for a system in Fig. 7,*a* are not evident and result from a simultaneous solution of mutual systems.

5. Five electrodes in heterogeneous medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$. The system has a center of fivefold rotation symmetry and/or an axis (axes) of anti-symmetry or a homogeneous boundary with a center of tenfold rotational symmetry.

An example of respective two-component systems is given in Fig. 8,*a, b*; white domains have a dielectric permeability ε_M and grey domains ε_N.

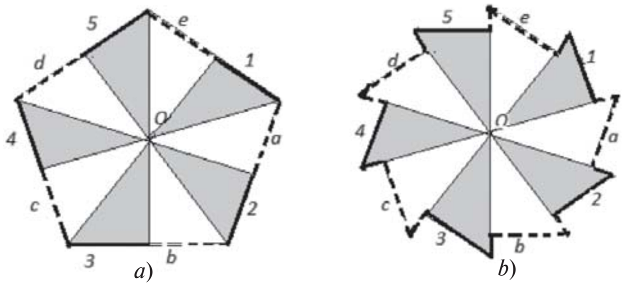


Fig. 8

Unit-length partial mutual capacitances (analytic solution of a system of five linear equations):

$$C_{12} = 4\sqrt{\varepsilon_M \varepsilon_N} / \sqrt{50+10\sqrt{5}} = 4K / \sqrt{50+10\sqrt{5}};$$

$$C_{12} = C_{23} = C_{34} = C_{45} = C_{15};$$

$$C_{13} = (\sqrt{5}-1)\sqrt{\varepsilon_M \varepsilon_N} / \sqrt{50+10\sqrt{5}} =$$

$$= (\sqrt{5}-1)K / \sqrt{50+10\sqrt{5}};$$

$$C_{13} = C_{14} = C_{24} = C_{25} = C_{35}.$$

6*a*. Six electrodes in heterogeneous medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$. The system has a center of 6-fold rotational symmetry and/ or an axis (axes) of anti-symmetry or a homogeneous boundary with a center of 12-fold rotational symmetry.

Figures illustrating these examples are similar to those given for a problem with five electrodes.

Unit-length partial mutual capacitances (analytic solution of a system of six linear equations):

$$C_{12} = (1+\sqrt{3})\sqrt{\varepsilon_M \varepsilon_N} / 6 = (1+\sqrt{3})K / 6;$$

$$C_{12} = C_{23} = C_{34} = C_{45} = C_{16};$$

$$C_{13} = (\sqrt{3}-1)\sqrt{\varepsilon_M \varepsilon_N} / 6 = (\sqrt{3}-1)K / 6;$$

$$C_{13} = C_{24} = C_{35} = C_{46};$$

$$C_{14} = (2-\sqrt{3})\sqrt{\varepsilon_M \varepsilon_N} / 3 = (2-\sqrt{3})K / 3;$$

$$C_{14} = C_{25} = C_{36}.$$

6*b*. Six electrodes in heterogeneous medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$. The system has an axis (axes) of anti-symmetry and a center of 2-fold rotational symmetry. A homogeneous system boundary has a center of 4-fold rotational symmetry.

Examples of respective two-component systems are given in Fig. 9,*a, b*; white domains have a dielectric permeability ε_M and grey domains ε_N.

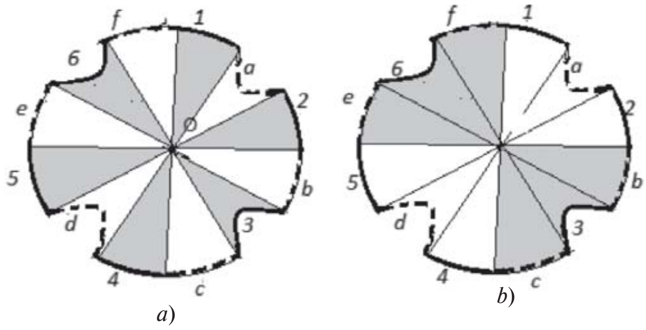


Fig. 9

All solutions for mutual partial capacitances are similar to those of problem 6*a*.

7. Seven electrodes in heterogeneous two-dimensional medium. The system can be reduced to a matching mutual system, $\sqrt{\varepsilon_M \varepsilon_N} = K$. The system has a center of 7-fold rotational symmetry and/ or an axis (axes) of anti-symmetry or a homogeneous boundary with a center of 14-fold rotational symmetry.

Figures illustrating these examples are similar to those given for a problem with 5 and 6 electrodes respectively.

Unit-length partial mutual capacitances (analytic solution of a system of seven linear equations):

$$C_{12} = 0.442\sqrt{\varepsilon_M \varepsilon_N} = 0.442K;$$

$$C_{12} = C_{23} = C_{34} = C_{45} = C_{56} = C_{17};$$

$$C_{13} = 0.112\sqrt{\varepsilon_M \varepsilon_N} = 0.112K;$$

$$C_{13} = C_{24} = C_{35} = C_{46} = C_{57};$$

$$C_{14} = 0.072\sqrt{\varepsilon_M \varepsilon_N} = 0.072K;$$

$$C_{14} = C_{25} = C_{36} = C_{47}.$$

Here it would be appropriate to compare all the solutions obtained above for mutual partial capacitances of studied symmetric multi-electrode systems with the data of mathematical theory of regular *n*-gon. In particular, expressions of radius of circumscribed and inscribed circles and regular *n*-gon area for $n = 3, 4, 5, 6$ have a structure and contain radicals which are similar to solutions for respectively 3-, 4-, 5- and 6-electrode systems while there is no analytic solution for a regular septagon (7-gon) and it seems likely that this is due to the impossibility to build such a polygon by means of compasses and ruler.

Let us compare e.g. expressions for mutual partial capacitance of neighboring electrodes at $K = 1$ for considered 5-electrode system

$$C_{12} = 4 / \sqrt{50+10\sqrt{5}}$$

and for ratio of side length l and circumscribed circle radius r_e in the case of regular pentagon

$$I / r_e = 10 / \sqrt{50 + 10\sqrt{5}}.$$

The work on finding solutions for systems with $n > 7$ could be continued and, based on conclusions of regular gon theory, one should expect only computational solutions to exist for $n = 9$, 11, 13, 18, 19, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 33, ...

The following common rule (theorem) may be formulated on the basis of obtained results:

If a two-dimensional system of alternating n electrodes and n impermeable boundaries, which form together in transversal section a closed contour encircling a simply connected surface, is matching with its mutual system created by Keller—Dykhne inversion, and a number of system various mutual partial capacitances does not exceed n, then the values of all mutual partial capacitances are:

invariant in terms of system linear dimensions and boundary shapes;

directly proportional to geometric mean value of system dielectric permeability;

the sole body of $[n/2]$ positive real roots of a specified $[n/2]$ quadratic equations; $[n/2]$ is the integral part of number $n/2$.

Keller—Dykhne inversion for Thompson—Lampard theorem. The method of Keller—Dykhne inversion may be applied to analytic solution and calculation of parameters of a more extended class of multi-electrode two-dimensional systems, when a part of integral parameters of mutually inverted systems was obtained by other means, e.g. by numerical calculations of fields, by experiment or with the use of other analytic methods.

Let us consider as an example of such an application a known four electrode system shown on Fig. 10 which was solved following Thompson — Lampard theorem [7] for mutual partial capacitances of opposite electrodes. Infinite small linear interspaces, perpendicular to section planes (points a, b, c and d on Fig. 10) separate electrodes in in such a way that they may have different potentials.

For a system with an arbitrary in shape transversal section it has been proved that

$$e^{-\pi C_1/\varepsilon} + e^{-\pi C_2/\varepsilon} = 1,$$

ε — dielectric permeability of medium containing a system, C_1 and C_2 — mutual partial capacitances between electrodes ab-cd and ad-bc inside or outside a closed cylindrical shell containing electrodes and interspaces between them.

In case the system possesses a symmetry plane (ac on Fig. 11) or 4-fold rotational symmetry axis (center O on Fig. 12) the generalized theorem implies the following

$$C_1 = C_2 = \varepsilon \ln 2 / \pi.$$

Thompson—Lampard theorem has found its practical implementation in metrology during elaboration of capacitance standard.

Let electrode ab on Fig.11 and 12 have a potential U, the potential of three other electrodes is 0 and a surface linear charge of electrode cd (per cylinder unit-length — square to the surface of transversal section) is Q. A dielectric permeability of medium is ε. Then a unit-length partial mutual capacitance between electrodes ab and cd per unit length is $C = Q/U = = \varepsilon \ln 2 / \pi$.

The method of Keller—Dykhne inversion as applied to this problem (matched rotation of strength and displacement fields in each point by 90° around normal line with respect to transversal section, for example, clockwise) results in a new system of linear electrodes (point electrodes in transversal section) a, b, c, d and impermeable surfaces (lines in transversal section) ab, bc, cd, da (Fig.13).

The shape of surface boundaries is invariable and a

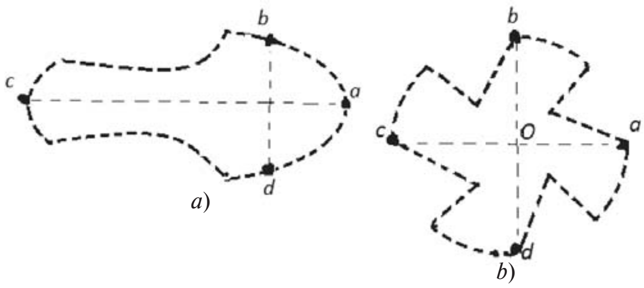


Fig. 13

dielectric permeability ε' inside and outside this surface should be equal to $1/\varepsilon$: $\varepsilon' = 1/\varepsilon$. In this case a linear charge Q' of electrode a is equal to voltage U (each vector E becomes vector D and vice versa), a linear charge Q' of electrode b is equal to voltage U, and the voltage U' between electrodes c and d is equal to Q, so we shall have relationships $\alpha U'/Q' = Q/U = \alpha = \ln 2/\varepsilon'\pi = \varepsilon \ln 2/\pi$ for inner and outer problems.

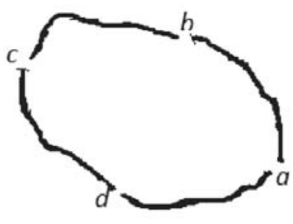


Fig. 10

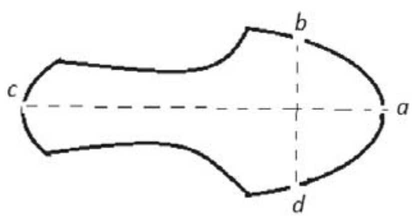


Fig. 11

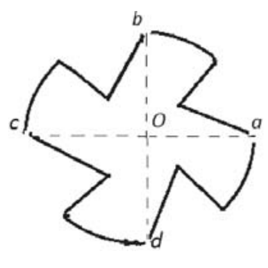
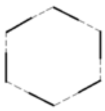


Fig. 12

Thus, a system of linear electrodes in medium e’ having an exact solution for potential factor which considers the voltage and charges in a system as a function of one parameter — dielectric permeability of medium - is associated with every two-dimensional system of surface electrodes in medium e having a definite exact solution for mutual partial capacitance between opposite electrodes following Thompson—Lampard theorem with the help of Keller—Dykhne inversion. Similar to the basic Thompson—Lampard theorem, this result may be implemented in practical metrology.

Appendix. Mutual partial linear capacitances of compatible two-dimensional inverse n-electrode systems (e^* - mean geometric value of medium dielectric, k — electrode index number)

Number of electrodes	Transversal section (sample)	$C_{k,k+1}/e^*$ $k > n$	$C_{k,k+2}/e^*$ $k < n-1$	$C_{k,k+3}/e^*$ $k < n-2$
2		1	–	–
3		$1/\sqrt{3}$	–	–
4		1/2	$(\sqrt{2}-1)/2$	–
5		$4/\sqrt{50+10\sqrt{5}}$	$(\sqrt{5}-1)/\sqrt{50+10\sqrt{5}}$	–
6		$(1+\sqrt{3})/6$	$(\sqrt{3}-1)/6$	$(2-\sqrt{3})/3$
7		0.442	0.112	0.072

LIST OF REFERENCES

1. **Keller J.B.** A Theorem on the Conductivity of a Composite Medium. — Journal of Mathematical Physics, 1964, vol. 5, No. 3, pp. 548—549.

2. **Dykhne A.M.** Conductivity of two-dimensional two-phase system. Journal book of experimental and theoretical physics, 1970, vol. 59, iss. 1(7), pp. 63—65.

3. **Klyamkin S.S.** Efficient conductivity of symmetrical two-dimensional heterogeneous systems and electrode systems placed therein. — Proceedings of the USSR Academy of Sciences. Energy and transport, 1988, №. 2, pp. 164—169.

4. **Klyamkin S.S.** Efficient conductivity of composite materials with symmetrical structure. — Proceedings of HVDC Power Transmission Research Institute, 2006, №. 61, pp. 83—110.

5. **Snarsky A., Sliptchenko K., Palti A.** Electrical samples. — Kvant, 2009, №. 2, pp. 2—5.

6. **Iossel Yu. Ya., Kochanov E.S., Strunsky M.G.** Electrical capacitance calculation. — L.: Energoizdat, 1981, 288 p.

7. **Jackson J.D.** A curious and useful theorem in two-dimensional electrostatics. — Am. J. Phys, 1999, 67 (2), pp. 107—115.

[03.09.15]

Электричество, 20016, No. 1, с. 35—43.

Использование метода инверсии Келлера—Дихне для определения интегральных параметров мультиэлектродных двухмерных систем

КЛЯМКИН С.С.

Предложен новый метод расчета интегральных параметров двухмерных многоэлектродных устройств, имеющих оси или плоскости симметрии (электрической проводимости, емкости и др.).

Метод основан на совместном решении алгебраических уравнений для исходной и инверсной по отношению к ней систем (устройств), связанных между собой по принципу поворотной инвариантности двухмерных потенциальных и соленоидальных полей. Полученные на основе метода неизвестные ранее формулы для расчета погонной электрической емкости устройств сведены в таблицу, приведенную в приложении. Метод позволил расширить область применения теоремы Томпсона—Лэмпарда (Thomson—Lampard) взаимной частичной емкости на единицу длины в системе четырех цилиндрических проводящих пластин, имеющих плоскость симметрии. Полученные на основе теоремы Томпсона—Лэмпарда решения можно использовать для более широкого круга задач электротехники. Найденная связь между точными решениями для симметричных многоэлектродных систем и геометрическими параметрами правильных многоугольников позволяет исследовать свойства подобных систем без выполнения расчетов. Предложенный метод можно применить для расчета не только погонной емкости, но и других интегральных параметров двухмерных физических систем при различных условиях симметрии и распределения характеристик сред.

Ключевые слова: *многоэлектродные устройства, электрическая проводимость, емкость, двухмерные физические системы, определение интегральных параметров, теорема Томпсона—Лэмпарда*

Автор: ***С.С. Клямкин*** *окончил Ленинградский политехнический институт в 1962 г. В 1970 г. защитил кандидатскую диссертацию «Исследование электрических характеристик многопроволочных проводов на звуковых частотах для высоковольтных линий электропередачи постоянного тока». Научный сотрудник (Вернон, Калифорния, США).*